Weighted Endpoint Estimates for Multilinear Commutators of Marcinkiewicz Integrals

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Abstract: Let $\mu_{\Omega,\vec{b}}$ be the multilinear commutator generalized by μ_{Ω}, the n-dimensional Marcinkiewicz integral, with $\mathrm{Osc}_{\exp L^{\tau}}(\mathbb{R}^n)$ functions for $\tau \geq 1$, where $\mathrm{Osc}_{\exp L^{\tau}}(\mathbb{R}^n)$ is a space of Orlicz type satisfying that $\mathrm{Osc}_{\exp L^{\tau}}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ if $\tau = 1$ and $\mathrm{Osc}_{\exp L^{\tau}}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ if $\tau > 1$. The authors establish the weighted weak $L \log L$-type estimates for $\mu_{\Omega,\vec{b}}$ when Ω satisfies a kind of Dini conditions.

Keywords: Marcinkiewicz integral; multilinear commutator; A_p weight; Orlicz space

MR(2000) Subject Classification: 42B20; 42B25; 42B99

1   Introduction and Main Result

Denote by S^{n-1} the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $\mathrm{d}x' = \mathrm{d}\sigma(x')$. Let $\Omega(x) \in L^1(S^{n-1})$ be homogeneous function of degree zero in $\mathbb{R}^n$ satisfying

$$\int_{S^{n-1}} \Omega(x')\mathrm{d}x' = 0, \tag{1.1}$$

where $x' = x/|x|$ $(x \neq 0)$.

The n-dimensional Marcinkiewicz integral introduced by Stein [1] is defined by

$$\mu_{\Omega}(f)(x) = \Big(\int_0^{\infty} \Big| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

A weight will always means a positive locally integrable function. As usual, we denote by A_p $(1 \leq p \leq \infty)$ the Muckenhoupt weights classes (see [2] and [3] for details). For a weight ω on $\mathbb{R}^n$, we write $\|f\|_{L^p_{\omega}(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\mathrm{d}x)^{1/p}$ and $\omega(E) = \int_E \omega(x)\mathrm{d}x$.

In 2004, Ding, Lu and Zhang [4] studied the weighted weak $L \log L$-type estimates for the commutators of the Marcinkiewicz integral, which is defined by

$$\mu^m_{\Omega,b}(f)(x) = \Big(\int_0^{\infty} \Big| \int_{|x-y| \leq t} \frac{(b(x) - b(y))^m \Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}}, \quad m \in \mathbb{Z}^+, b \in \mathrm{BMO}(\mathbb{R}^n),$$

when the kernel Ω satisfies the $\mathrm{Lip}_{\alpha}(0 < \alpha \leq 1)$ condition, that is, there exists a constant $C > 0$ such that

$$|\Omega(x') - \Omega(y')| \leq C|x' - y'|^{\alpha}, \quad \forall x', y' \in S^{n-1}. \tag{1.2}$$

In 2008, Zhang [5] established the weighted weak $L(\log L)^{1/r}$-type estimates for the multilinear commutators of the Marcinkiewicz integral when $\omega \in A_1$, and Ω satisfies (1.1) and (1.2).

Let $\Omega \in L^r(S^{n-1})$ $(r \geq 1)$, the integral modulus of continuity of order r of Ω is defined by

$$\omega_r(\delta) = \sup_{|\rho| < \delta} \left(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^r \mathrm{d}x' \right)^{1/r},$$

where ρ is a rotation in $\mathbb{R}^n$ with $|\rho| = \sup_{x' \in S^{n-1}} |\rho x' - x'|$.

We say $\Omega \in L^r(S^{n-1})$ $(r \geq 1)$ satisfies the L^r-Dini condition if $\int_0^1 \omega_r(\delta)\delta^{-1}\mathrm{d}\delta < \infty$.

Recently, Zhang [6] also considered the following result.

Theorem A. [6] Let $b \in \mathrm{BMO}(\mathbb{R}^n), \Omega \in L^r(S^{n-1})$ for some $r > 1$, and $\omega^{r'} \in A_1$. If Ω satisfies (1.1) and

$$\int_0^1 \frac{\omega_r(\delta)}{\delta} \left(\log \frac{1}{\delta} \right)^m \mathrm{d}\delta < \infty, \tag{1.3}$$

then for all $\lambda > 0$, there has

$$\omega(\{x \in \mathbb{R}^n : \mu_{\Omega,b}^m(f)(x) > \lambda\}) \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \frac{|f(y)|}{\lambda} \right)^m \omega(y)\mathrm{d}y,$$

where C is a positive constant independent of f and λ.

In this paper, by applying the calderón-Zygmund decomposition theory, we will study the weighted weak $L \log L$-type estimates for the multilinear commutators generated by μ_Ω and $\mathrm{Osc}_{\exp L^r}(\mathbb{R}^n)$ functions, in analogy with the results established by Pérez and Trujillo-González in [7] for the multilinear commutators of Calderón-Zygmund operators. Before stating our results, we first recall some notation.

Let m be a positive integer and $\vec{b} = (b_1, b_2, \cdots, b_m)$, we define the multilinear commutators $\mu_{\Omega,\vec{b}}$ by

$$\mu_{\Omega,\vec{b}}(f)(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)f(y)}{|x-y|^{n-1}} \prod_{j=1}^m \left(b_j(x) - b_j(y) \right)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{\frac{1}{2}}.$$

It is easy to see, when $m = 1$, $\mu_{\Omega,\vec{b}}$ is the commutator of Marcinkiewicz integral and when $b_1 = \cdots = b_m$, $\mu_{\Omega,\vec{b}}$ is the higher commutator of Marcinkiewicz integral.

To state the weak type estimate for the multilinear commutator $\mu_{\Omega,\vec{b}}$, we need to introduce the following notation. As in [7], given any positive integer m, for all $1 \leq j \leq m$, we denote by $\mathscr{C}_j^m$ the family of all finite subsets $\sigma = \{\sigma(1), \sigma(2), \cdots, \sigma(j)\}$ of $\{1, 2, \cdots, m\}$ with j different elements. For any $\sigma \in \mathscr{C}_j^m$, we define the complementary sequence $\sigma' = \{1, 2, \cdots, m\} \setminus \sigma$.

In the following, we will always assume that Ω be homogeneous function of degree 0, and let $\vec{b} = (b_1, b_2, \cdots, b_m)$ be a finite family of locally integrable functions. For all $1 \leq j \leq m$ and $\sigma = \{\sigma(1), \sigma(2), \cdots, \sigma(j)\} \in \mathscr{C}_j^m$, we write for any i-tuple $(\tau_1, \tau_2, \cdots, \tau_m)$ with $\tau_j \geq 1$ for $1 \leq j \leq m$, $1/\tau_\sigma = 1/\tau_{\sigma(1)} + \cdots + 1/\tau_{\sigma(j)}$ and $1/\tau_{\sigma'} = 1/\tau - 1/\tau_\sigma$, where $1/\tau = 1/\tau_1 + \cdots + 1/\tau_m$, we will denote $\vec{b}_\sigma = (b_{\sigma(1)}, b_{\sigma(2)}, \cdots, b_{\sigma(j)})$ and the product $b_\sigma = b_{\sigma(1)}b_{\sigma(2)} \cdots b_{\sigma(j)}$. With this notation, we write

$$\|\vec{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{\tau_\sigma}}(\mathbb{R}^n)} = \|b_{\sigma(1)}\|_{\mathrm{Osc}_{\exp L^{\tau_{\sigma(1)}}}(\mathbb{R}^n)} \cdots \|b_{\sigma(j)}\|_{\mathrm{Osc}_{\exp L^{\tau_{\sigma(j)}}}(\mathbb{R}^n)}.$$

In particular, we write

$$\left(b(x) - b(y) \right)_\sigma = \left(b_{\sigma(1)}(x) - b_{\sigma(1)}(y) \right) \cdots \left(b_{\sigma(j)}(x) - b_{\sigma(j)}(y) \right),$$

and

$$\left(b_B - b(y) \right)_\sigma = \left((b_{\sigma(1)})_B - b_{\sigma(1)}(y) \right) \cdots \left((b_{\sigma(j)})_B - b_{\sigma(j)}(y) \right),$$

where B is any ball in $\mathbb{R}^n$, $x, y \in \mathbb{R}^n$, and $f_B = \frac{1}{|B|} \int_B f(y)\mathrm{d}y$. For any $\sigma \in \mathscr{C}_j^m$, we set

$$\mu_{\Omega,\vec{b}_\sigma}(f)(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)f(y)}{|x-y|^{n-1}} \prod_{i=1}^j \left(b_{\sigma(i)}(x) - b_{\sigma(i)}(y) \right)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{\frac{1}{2}}.$$

If $\sigma = \{1, 2, \cdots, m\}$, then σ' is an empty set, we understand $\mu_{\Omega, \vec{b}_\sigma} = \mu_{\Omega, \vec{b}}$ and $\mu_{\Omega, \vec{b}_{\sigma'}} = \mu_\Omega$.

Our result can be stated as follows.

Theorem 1.1. Let $b_j \in \mathrm{Osc}_{\exp L^{\tau_j}}, \tau_j \geq 1 \ (1 \leq j \leq m), \Omega \in L^r(S^{n-1})$ for some $r > 1$, and $\omega^{r'} \in A_1$. If Ω satisfies (1.1) and (1.3), then for all $\lambda > 0$, there has

$$\omega(\{x \in \mathbb{R}^n : \mu_{\Omega, \vec{b}}(f)(x) > \lambda\}) \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \Big(1 + \log^+ \frac{|f(y)|}{\lambda}\Big)^m \omega(y) dy,$$

where C is a positive constant independent of f and λ.

Remark 1. Noting that $\mathrm{Osc}_{\exp L^1} = \mathrm{BMO}$ and $\mathrm{Osc}_{\exp L^\tau} \subset \mathrm{BMO}$ for $\tau > 1$. For more information on Orlicz space see [10].

Obviously, condition (1.3) is slightly stronger than the L^r-Dini condition, but much more weaker than the Lip_α condition. Noting that $\mu_{\Omega, \vec{b}}$ coincides with $\mu_{\Omega, b}^m$ when $b_j = b$ for $j = 1, 2, \cdots, m$. So, Theorem 1.1 improves the main results in [5] and [6].

Throughout this paper, C denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$. For $A \sim B$, we mean that there is a constant $C > 0$ such that $C^{-1}B \leq A \leq CB$.

2 Preliminaries and Lemmas

In this section, we will formulate some lemmas and preliminaries.

Lemma 2.1. [8] Suppose that $0 < \alpha < n, r > 1$ and Ω satisfies the L^r-Dini condition. If there is a constant C_0 with $0 < C_0 < 1/2$ such that $|y| < C_0 K$, then

$$\Big(\int_{K<|x|<2K} \Big| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}} \Big|^r dx\Big)^{1/r} \leq CK^{n/r-n+\alpha} \Big(\frac{|y|}{K} + \int_{|y|/(2K)<\delta<|y|/K} \frac{\omega_r(\delta)}{\delta} d\delta\Big).$$

Lemma 2.2. [9] Suppose $\Omega \in L^r(S^{n-1})$ for some $r > 1$ and $\omega^{r'} \in A_1$. Then for any $\lambda > 0$, there is a constant $C > 0$ independent of f and λ, such that

$$\omega(\{x \in \mathbb{R}^n : \mu_\Omega(f)(x) > \lambda\}) \leq C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}.$$

Lemma 2.3. Let $\omega \in A_1, 1 < p < \infty, b_j \in \mathrm{Osc}_{\exp L^{\tau_j}}, \ \tau_j \geq 1 \ (1 \leq j \leq m), \Omega \in L^r(S^{n-1})$ for some $r > 1$ and satisfies (1.1) and (1.3). Then, there is a constant $C > 0$ independent of f, such that

$$\|\mu_{\Omega, \vec{b}}(f)\|_{L^p_\omega(\mathbb{R}^n)} \leq C\|\vec{b}\|_{\mathrm{Osc}_{\exp L^\tau}}\|f\|_{L^p_\omega(\mathbb{R}^n)}.$$

The idea of the proof of Lemma 2.3 comes from the corollary 1.3 in [5]. We omit the details.

We also need a few facts of Orlicz spaces, see [10] for more information.

A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called a Young function if φ is continuous, convex and increasing with $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to +\infty$. We defined the φ-average of a function f over a ball B by means of the Luxemburg norm

$$\|f\|_{\varphi, B} = \inf \Big\{\lambda > 0 : \frac{1}{|B|} \int_B \varphi\Big(\frac{|f(y)|}{\lambda}\Big) dy \leq 1\Big\},$$

which satisfies the following inequalities (see [10], P.69 or formula (7) in [11])

$$\|f\|_{\varphi, B} \leq \inf \Big\{\eta + \frac{\eta}{|B|} \int_B \varphi\Big(\frac{|f(y)|}{\eta}\Big) dy \leq 1\Big\} \leq 2\|f\|_{\varphi, B}. \tag{2.1}$$

The Young function that we are going to be using is $\Phi_\alpha(t) = t(1 + \log^+ t)^\alpha$ ($\alpha > 0$) with its complementary Young function $\tilde{\Phi}_\alpha(t) \approx \exp(t^{1/\alpha})$. Denote by $\|f\|_{L(\log L)^\alpha, B} = \|f\|_{\Phi_\alpha, B}$ and $\|f\|_{\exp L^{1/\alpha}, B} = \|f\|_{\tilde{\Phi}_\alpha, B}$. When $\alpha = 1$, we simply denote by $\Phi(t) = t(1 + \log^+ t)$ and $\tilde{\Phi}(t) \approx e^t$, and by $\|f\|_{L \log L, B} = \|f\|_{\Phi, B}$ and $\|f\|_{\exp L, B} = \|f\|_{\tilde{\Phi}, B}$. By the generalized Hölder's inequality (see [12]), we have

$$\frac{1}{|B|} \int_B |f(y)g(y)| \mathrm{d}y \leq 2\|f\|_{L(\log L)^\alpha, B} \|g\|_{\exp L^{1/\alpha}, B}. \tag{2.2}$$

As usual, for a locally integrable function f and a ball B, we denote $f_B = \frac{1}{|B|} \int_B f(y)\mathrm{d}y$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, for any ball B and integer $k \geq 0$, there has (see [2], p.141)

$$|b_{2^{k+1}B} - b_B| \leq C(k+1)\|b\|_*, \tag{2.3}$$

where ℓB denotes the ℓ-times concentric expansion of B and $\|b\|_*$ denotes the BMO norm of b.

By the John-Nirenberg's inequality, it is not difficult to see that (c.f. [13], p.169)

$$\|b - b_B\|_{\exp L, B} \leq C\|b\|_*. \tag{2.4}$$

Let $M_{L(\log L)^\alpha}(f)(x) = \sup_{B \ni x} \|f\|_{L(\log L)^\alpha, B}$. Denote by M the Hardy-Littlewood maximal function and M^k the k-times iterations of M, then $M_{L(\log L)^k} \approx M^{k+1}$ for $k = 0, 1, 2, \cdots$. We also need the following estimates in the proof of Theorem 1.1.

Lemma 2.4. [5] Let $1 \leq p < \infty, \omega^p \in A_1$ and B be a ball. Then for any $y \in B$ and any positive integer m, there has

$$\left(\frac{1}{|2^k B|} \int_{2^k B} |b(x) - b_B|^{mp} \omega^p(x)\mathrm{d}x\right)^{1/p} \leq C\|b\|_*^m (k+1)^m \inf_{y \in B} \omega(y), \ k = 0, 1, 2, \cdots.$$

Lemma 2.5. Let $1 \leq p < \infty, \omega^p \in A_1$ and B be a ball. Then for any $y \in B$ and any positive integer m, there has

$$\left(\frac{1}{|2^k B|} \int_{2^k B} \omega^p(x) \prod_{j=1}^m \left|b_j(x) - (b_j)_B\right|^p \mathrm{d}x\right)^{1/p} \leq C\|\vec{b}\|_* (k+1)^m \inf_{y \in B} \omega(y), \ k = 0, 1, 2, \cdots.$$

Proof. By the Hölder's inequality and Lemma 2.4, we obtain

$$
\begin{aligned}
\left(\frac{1}{|2^k B|} \int_{2^k B} \omega^p(x) \prod_{j=1}^m \left|b_j(x) - (b_j)_B\right|^p \mathrm{d}x\right)^{1/p} &\leq \prod_{j=1}^m \left(\frac{1}{|2^k B|} \int_{2^k B} \omega^p(x) \left|b_j(x) - (b_j)_B\right|^{p\gamma_j} \mathrm{d}x\right)^{\frac{1}{p\gamma_j}} \\
&\leq C \prod_{j=1}^m \left(\|b_j\|_*^{\gamma_j} (k+1)^{\gamma_j} \inf_{y \in B} \omega(y)\right)^{\frac{1}{\gamma_j}} \\
&\leq C\|\vec{b}\|_* (k+1)^m \inf_{y \in B} \omega(y),
\end{aligned}
$$

where $1 = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \cdots + \frac{1}{\gamma_m}$.

This completes the proof of Lemma 2.5. □

We also need the following notations. For $\omega \in A_\infty$ and a ball B, denote by

$$\|f\|_{L(\log L)^m, B, \omega} = \inf\left\{\lambda > 0 : \frac{1}{\omega(B)} \int_B \Phi_m\left(\frac{|f(y)|}{\lambda}\right) \omega(y)\mathrm{d}y \leq 1\right\}$$

and

$$\|f\|_{\exp L^{1/m}, B, \omega} = \inf\left\{\lambda > 0 : \frac{1}{\omega(B)} \int_B \tilde{\Phi}_m\left(\frac{|f(y)|}{\lambda}\right) \omega(y)\mathrm{d}y \leq 1\right\}.$$

Similar to (2.1), we have (c.f. [10], p.69)

$$\|f\|_{L(\log L)^m, B, \omega} \approx \inf \left\{ \eta + \frac{\eta}{\omega(B)} \int_B \Phi_m \left(\frac{|f(y)|}{\eta} \right) \omega(y) \mathrm{d}y \right\}. \tag{2.5}$$

By (2.2), there also holds the following generalized Hölder's inequality

$$\frac{1}{\omega(B)} \int_B |f_1(y) \cdots f_m(y) g(y)| \omega(y) \mathrm{d}y \leq C \|g\|_{L(\log L)^m, B, \omega} \prod_{j=1}^m \|f_j\|_{\exp L, B, \omega}. \tag{2.6}$$

Furthermore, for any $b \in \mathrm{BMO}(\mathbb{R}^n)$, any ball B and any $\omega \in A_\infty$, there has

$$\|b - b_B\|_{\exp L, B, \omega} \leq C \|b\|_*, \tag{2.7}$$

Indeed, by John-Nirenberg's inequality, there exist positive constants C_1 and C_2, such that

$$|\{x \in B : |b(x) - b_B| > t\}| \leq C_1 |B| e^{-C_2 t / \|b\|_*}.$$

Noting that $\omega \in A_\infty$, from the proof of Theorem 5 in [14], there is a $\delta > 0$, such that

$$\omega(\{x \in B : |b(x) - b_B| > t\}) \leq C_1 \omega(B) e^{-C_2 \delta t / \|b\|_*}.$$

Similar to the proof of Corollary 7.1.7 in [3] (p.528), we have

$$\frac{1}{\omega(B)} \int_B \exp \left(\frac{|b(x) - b_B|}{C_3 \|b\|_*} \right) \omega(x) \mathrm{d}x \leq C, \tag{2.8}$$

which implies (2.7).

3 Proof of Theorem1.1

Proof. Without loss of generality, we may assume that for $j = 1, \ldots, m, \|b_j\|_{\mathrm{Osc}_{\exp L^{\tau_j}}(\mathbb{R}^n)} = 1$. In fact, let

$$\tilde{b}_j = \frac{b_j}{\|b_j\|_{\mathrm{Osc}_{\exp L^{\tau_j}}(\mathbb{R}^n)}}$$

for $j = 1, \ldots, m$. The homogeneity tells us that for any $\lambda > 0$,

$$\omega(\{x \in \mathbb{R}^n : \mu_{\Omega, \vec{b}}(f)(x) > \lambda\}) = \omega(\{x \in \mathbb{R}^n : \mu_{\Omega, \tilde{b}}(f)(x) > \lambda / \|\vec{b}\|_{\mathrm{Osc}_{\exp L^\tau}(\mathbb{R}^n)}\}) \tag{3.1}$$

Noting that $\|\tilde{b}_j\|_{\mathrm{Osc}_{\exp L^{\tau_j}}(\mathbb{R}^n)} = 1$ for $j = 1, \ldots, m$, if when $\|b_j\|_{\mathrm{Osc}_{\exp L^{\tau_j}}(\mathbb{R}^n)} = 1$ $(j = 1, \ldots, m)$, the theorem is true. , by (3.1) and the inequality

$$\Phi_s(t_1 t_2) \leq C \Phi_s(t_1) \Phi_s(t_2)$$

for any $s > 0, t_1, t_2 \geq 0$, we easily obtain that the theorem still holds for any $b_j \in \mathrm{Osc}_{\exp L^{\tau_j}}(\mathbb{R}^n)$ $(j = 1, \ldots, m)$.

For a fixed λ, we consider the Calderón-Zygmund decomposition of f at height λ and get a sequence of balls $\{B_i\}$, where B_i is a ball centered at x_i with radius r_i, such that $|f(x)| \leq C\lambda$ for *a.e.* $x \in \mathbb{R}^n \setminus \cup_i B_i$ and

$$\lambda < \frac{1}{|B_i|} \int_{B_i} |f(y)| \mathrm{d}y \leq 2^n \lambda. \tag{3.2}$$

Moreover, there is an integer $N \geq 1$, independent of f and λ, such that for every point in $\mathbb{R}^n$ belongs to at most N balls in $\{B_i\}$.

We decompose $f = g + h$, where

$$g(x) = \begin{cases} f(x), & x \in \mathbb{R}^n \setminus \cup_i B_i, \\ f_{B_i}, & x \in B_i. \end{cases}$$

Then $h(x) = f(x) - g(x) = \sum_i h_i(x)$ with $h_i(x) = (f(x) - f_{B_i})\chi_{B_i}(x)$. Obviously, $\operatorname{supp} h_i \subset B_i$, $\int_{B_i} h_i(y)\mathrm{d}y = 0$ and

$$|g(x)| \leq 2^n \lambda, \quad a.e. \ x \in \mathbb{R}^n. \tag{3.3}$$

Noting that if $\omega^{r'} \in A_1$ then $\omega \in A_1$, and then $M(\omega)(x) \leq C\omega(x)$ for $a.e.\ x \in \mathbb{R}^n$. By (3.2) and the fact that $|B_i|^{-1}\omega(B_i) = |B_i|^{-1}\int_{B_i}\omega(x)\mathrm{d}x \leq C \inf_{y \in B_i}\omega(y)$, we have

$$\omega(B_i) \leq C|B_i| \inf_{y \in B_i} \omega(y) \leq C\lambda^{-1}\int_{B_i}|f(y)|\mathrm{d}y \inf_{y \in B_i}\omega(y) \leq C\lambda^{-1}\int_{B_i}|f(y)|\omega(y)\mathrm{d}y. \tag{3.4}$$

Denote by $E = \cup_i(4B_i)$, it follows from (3.4) that

$$\omega(E) \leq C \sum_i \int_{B_i}\omega(x)\mathrm{d}x = C \sum_i \omega(B_i) \leq C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}.$$

Write

$$\omega(\{x \in \mathbb{R}^n : \mu_{\Omega,\vec{b}}(f)(x) > \lambda\}) \leq \omega(\{x \in \mathbb{R}^n \setminus E : \mu_{\Omega,\vec{b}}(f)(x) > \lambda\}) + \omega(E)$$

$$\leq \omega(\{x \in \mathbb{R}^n \setminus E : \mu_{\Omega,\vec{b}}(g)(x) > \frac{\lambda}{2}\}) + \omega(\{x \in \mathbb{R}^n \setminus E : \mu_{\Omega,\vec{b}}(h)(x) > \frac{\lambda}{2}\}) + \omega(E)$$

$$\leq I_1 + I_2 + C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}.$$

We consider I_1 first. For $\omega^{r'} \in A_1$ there has $\omega \in A_1$. Noting that $A_1 \subset A_s$ ($s \geq 1$), then for any $p > r'$, we have $\omega \in A_{p/r'}$. It follows from Lemma 2.3, (3.3) and (3.4) that

$$
\begin{aligned}
I_1 &\leq C\lambda^{-p}\int_{\mathbb{R}^n}\left(\mu_{\Omega,\vec{b}}(g)(x)\right)^p \omega(x)\mathrm{d}x \leq C\lambda^{-p}\int_{\mathbb{R}^n}|g(x)|^p\omega(x)\mathrm{d}x \leq C\lambda^{-1}\int_{\mathbb{R}^n}|g(x)|\omega(x)\mathrm{d}x \\
&\leq C\lambda^{-1}\left(\int_{\mathbb{R}^n \setminus \cup_i B_i}|g(x)|\omega(x)\mathrm{d}x + \int_{\cup_i B_i}|g(x)|\omega(x)\mathrm{d}x\right) \\
&\leq C\lambda^{-1}\left(\int_{\mathbb{R}^n}|f(x)|\omega(x)\mathrm{d}x + \sum_i \int_{B_i}|f_{B_i}|\omega(x)\mathrm{d}x\right) \\
&\leq C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)} + C\lambda^{-1}\sum_i \int_{B_i}\left(|B_i|^{-1}\int_{B_i}|f(y)|\mathrm{d}y\right)\omega(x)\mathrm{d}x \\
&\leq C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)} + C\lambda^{-1}\sum_i \int_{B_i}|f(y)|\mathrm{d}y\left(|B_i|^{-1}\int_{B_i}\omega(x)\mathrm{d}x\right) \\
&\leq C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)} + C\lambda^{-1}\sum_i \int_{B_i}|f(y)|\mathrm{d}y \inf_{y \in B_i}\omega(y) \\
&\leq C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)} + C\lambda^{-1}\sum_i \int_{B_i}|f(y)|\omega(y)\mathrm{d}y \\
&\leq C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}.
\end{aligned}
\tag{3.5}
$$

We remark that the proof of (3.5) implies the following fact, which will be used later.

$$\sum_i \int_{B_i}|f_{B_i}|\omega(x)\mathrm{d}x \leq C\|f\|_{L^1_\omega(\mathbb{R}^n)}. \tag{3.6}$$

Now, let us estimate I_2. By the definition of μ_Ω and $\mu_{\Omega,\vec{b}}$, with the aid of the formula

$$\prod_{j=1}^{m}\big(b_j(x)-b_j(y)\big)=\sum_{j=0}^{m}\sum_{\sigma\in\mathscr{C}_j^m}\Big(b(x)-b_{B_i}\Big)_\sigma\Big(b_{B_i}-b(y)\Big)_{\sigma'},$$

we have

$$\mu_{\Omega,\vec{b}}(h)(x)=\bigg(\int_0^\infty\bigg|\int_{|x-y|\le t}\frac{\Omega(x-y)h(y)}{|x-y|^{n-1}}\sum_{j=0}^{m}\sum_{\sigma\in\mathscr{C}_j^m}\Big(b(x)-b_{B_i}\Big)_\sigma\Big(b_{B_i}-b(y)\Big)_{\sigma'}\mathrm{d}y\bigg|^2\frac{\mathrm{d}t}{t^3}\bigg)^{\frac{1}{2}}$$

$$\le\bigg(\int_0^\infty\bigg|\int_{|x-y|\le t}\frac{\Omega(x-y)h(y)}{|x-y|^{n-1}}\prod_{j=1}^{m}\Big(b_j(x)-(b_j)_{B_i}\Big)\mathrm{d}y\bigg|^2\frac{\mathrm{d}t}{t^3}\bigg)^{\frac{1}{2}}$$

$$+\bigg(\int_0^\infty\bigg|\int_{|x-y|\le t}\frac{\Omega(x-y)h(y)}{|x-y|^{n-1}}\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\Big(b(x)-b_{B_i}\Big)_\sigma\Big(b_{B_i}-b(y)\Big)_{\sigma'}\mathrm{d}y\bigg|^2\frac{\mathrm{d}t}{t^3}\bigg)^{\frac{1}{2}}$$

$$+\bigg(\int_0^\infty\bigg|\int_{|x-y|\le t}\frac{\Omega(x-y)h(y)}{|x-y|^{n-1}}\prod_{j=1}^{m}\Big((b_j)_{B_i}-b_j(y)\Big)\mathrm{d}y\bigg|^2\frac{\mathrm{d}t}{t^3}\bigg)^{\frac{1}{2}}$$

$$\le\sum_i\prod_{j=1}^{m}\big|b_j(x)-(b_j)_{B_i}\big|\mu_\Omega(h_i)(x)+\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\Big|\big(b(x)-b_{B_i}\big)_\sigma\Big|\mu_\Omega\big(h_i\big(b_{B_i}-b\big)_{\sigma'}\big)(x)$$

$$+\mu_\Omega\Big(\sum_i h_i\prod_{j=1}^{m}\big((b_j)_{B_i}-b_j\big)\Big)(x).$$

So, we can write I_2 as

$$I_2\le\omega\Big(\Big\{x\in\mathbb{R}^n\setminus E:\sum_i\prod_{j=1}^{m}\big|b_j(x)-(b_j)_{B_i}\big|\mu_\Omega(h_i)(x)>\frac{\lambda}{6}\Big\}\Big)$$

$$+\omega\Big(\Big\{x\in\mathbb{R}^n\setminus E:\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\Big|\big(b(x)-b_{B_i}\big)_\sigma\Big|\mu_\Omega\big(h_i\big(b_{B_i}-b\big)_{\sigma'}\big)(x)>\frac{\lambda}{6}\Big\}\Big)\qquad(3.7)$$

$$+\omega\Big(\Big\{x\in\mathbb{R}^n\setminus E:\mu_\Omega\Big(\sum_i h_i\prod_{j=1}^{m}\big((b_j)_{B_i}-b_j\big)\Big)(x)>\frac{\lambda}{6}\Big\}\Big)$$

$$=I_{21}+I_{22}+I_{23}.$$

For I_{21}, using chebychev's inequality and Minkowski's inequality, we have

$$I_{21}=\omega\Big(\Big\{x\in\mathbb{R}^n\setminus E:\sum_i\prod_{j=1}^{m}\big|b_j(x)-(b_j)_{B_i}\big|\mu_\Omega(h_i)(x)>\frac{\lambda}{6}\Big\}\Big)$$

$$\le C\lambda^{-1}\sum_i\int_{\mathbb{R}^n\setminus 4B_i}\prod_{j=1}^{m}\big|b_j(x)-(b_j)_{B_i}\big|\mu_\Omega(h_i)(x)\omega(x)\mathrm{d}x$$

$$\le C\lambda^{-1}\sum_i\int_{\mathbb{R}^n\setminus 4B_i}\prod_{j=1}^{m}\big|b_j(x)-(b_j)_{B_i}\big|\bigg(\int_0^{|x-x_i|+2r_i}\bigg|\int_{|x-y|\le t}\frac{\Omega(x-y)h_i(y)}{|x-y|^{n-1}}\mathrm{d}y\bigg|^2\frac{\mathrm{d}t}{t^3}\bigg)^{\frac{1}{2}}\omega(x)\mathrm{d}x\qquad(3.8)$$

$$+C\lambda^{-1}\sum_i\int_{\mathbb{R}^n\setminus 4B_i}\prod_{j=1}^{m}\big|b_j(x)-(b_j)_{B_i}\big|\bigg(\int_{|x-x_i|+2r_i}^\infty\bigg|\int_{|x-y|\le t}\frac{\Omega(x-y)h_i(y)}{|x-y|^{n-1}}\mathrm{d}y\bigg|^2\frac{\mathrm{d}t}{t^3}\bigg)^{\frac{1}{2}}\omega(x)\mathrm{d}x$$

$$=I_{211}+I_{212}.$$

For $x\in\mathbb{R}^n\setminus 4B_i$ and $y\in B_i$, there has $|x-y|\le|x-x_i|+r_i$ and $|x-y|\sim|x-x_i|\sim|x-x_i|+2r_i$, and then

$$\int_{|x-y|}^{|x-x_i|+2r_i}\frac{\mathrm{d}t}{t^3}=\frac{1}{2}\Big(\frac{1}{|x-y|^2}-\frac{1}{(|x-x_i|+2r_i)^2}\Big)\le\frac{Cr_i}{|x-y|^3}.$$

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Noting that $\operatorname{supp} h_i \subset B_i$, it follows from the Minkowski's inequality that

$$I_{211} \leq C\lambda^{-1} \sum_i \int_{\mathbb{R}^n \backslash 4B_i} \prod_{j=1}^m \left| b_j(x) - (b_j)_{B_i} \right| \left(\int_{B_i} \frac{|\Omega(x-y)||h_i(y)|}{|x-y|^{n-1}} \left(\int_{|x-y|}^{|x-x_i|+2r_i} \frac{\mathrm{d}t}{t^3} \right)^{\frac{1}{2}} \mathrm{d}y \right) \omega(x) \mathrm{d}x$$

$$\leq C\lambda^{-1} \sum_i r_i^{1/2} \int_{\mathbb{R}^n \backslash 4B_i} \prod_{j=1}^m \left| b_j(x) - (b_j)_{B_i} \right| \left(\int_{B_i} \frac{|\Omega(x-y)||h_i(y)|}{|x-y|^{n+1/2}} \mathrm{d}y \right) \omega(x) \mathrm{d}x$$

$$\leq C\lambda^{-1} \sum_i r_i^{1/2} \int_{B_i} |h_i(y)| \sum_{k=1}^{\infty} \left(\int_{2^{k+1}B_i \backslash 2^k B_i} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}} \prod_{j=1}^m \left| b_j(x) - (b_j)_{B_i} \right| \omega(x) \mathrm{d}x \right) \mathrm{d}y$$

$$\leq C\lambda^{-1} \sum_i r_i^{1/2} \int_{B_i} |h_i(y)| \sum_{k=1}^{\infty} \left(\left(\int_{2^{k+1}B_i \backslash 2^k B_i} \frac{|\Omega(x-y)|^r}{|x-y|^{n+1/2}} \mathrm{d}x \right)^{1/r} \right.$$

$$\left. \times \left(\int_{2^{k+1}B_i \backslash 2^k B_i} \frac{\omega^{r'}(x)}{|x-y|^{n+1/2}} \prod_{j=1}^m \left| b_j(x) - (b_j)_{B_i} \right|^{r'} \mathrm{d}x \right)^{1/r'} \right) \mathrm{d}y$$

(3.9)

Noting that $2^{k-1}r_i \leq |x-y| \leq 2^{k+2}r_i$ whenever $y \in B_i$ and $x \in 2^{k+1}B_i \backslash 2^k B_i$, we have

$$\left(\int_{2^{k+1}B_i \backslash 2^k B_i} \frac{|\Omega(x-y)|^r}{|x-y|^{n+1/2}} \mathrm{d}x \right)^{1/r} \leq \left(\int_{2^{k-1}r_i \leq |x-y| \leq 2^{k+2}r_i} \frac{|\Omega(x-y)|^r}{|x-y|^{n+1/2}} \mathrm{d}x \right)^{1/r}$$

$$\leq \left(\int_{2^{k-1}r_i}^{2^{k+2}r_i} \rho^{n-1} \left(\int_{S^{n-1}} \frac{|\Omega(x')|^r}{\rho^{n+1/2}} \mathrm{d}x' \right) \mathrm{d}\rho \right)^{1/r} \leq C(2^k r_i)^{-\frac{1}{2r}} \|\Omega\|_{L^r(S^{n-1})}.$$

(3.10)

And noting that $\omega^{r'} \in A_1$ and $\|b_j\|_{\mathrm{BMO}} \leq C\|b_j\|_{\mathrm{Osc}_{\exp L^{\tau_j}}}$ for $\tau_j \geq 1$ ($1 \leq j \leq m$), by the Hölder's inequality, Minkowski's inequality, the properties of $\mathrm{BMO}(\mathbb{R}^n)$ functions and Lemma 2.5, we have

$$\left(\int_{2^{k+1}B_i \backslash 2^k B_i} \frac{\omega^{r'}(x)}{|x-y|^{n+1/2}} \prod_{j=1}^m \left| b_j(x) - (b_j)_{B_i} \right|^{r'} \mathrm{d}x \right)^{1/r'}$$

$$\leq C(2^{k+1}r_i)^{-(n+\frac{1}{2})/r'} \left(\int_{2^{k+1}B_i} \omega^{r'}(x) \prod_{j=1}^m \left| b_j(x) - (b_j)_{B_i} \right|^{r'} \mathrm{d}x \right)^{1/r'}$$

$$\leq C(2^{k+1}r_i)^{-(n+\frac{1}{2})/r'} \left(\frac{|2^{k+1}B_i|}{|2^{k+1}B_i|} \int_{2^{k+1}B_i} \omega^{r'}(x) \prod_{j=1}^m \left| b_j(x) - (b_j)_{B_i} \right|^{r'} \mathrm{d}x \right)^{1/r'}$$

$$\leq C(2^k r_i)^{-\frac{1}{2r'}} (k+1)^m \inf_{y \in B_i} \omega(y).$$

(3.11)

This, together with (3.9) and (3.10), gives

$$I_{211} \leq C\|\Omega\|_{L^r(S^{n-1})} \lambda^{-1} \sum_i r_i^{1/2} \int_{B_i} |h_i(y)| \left(\sum_{k=1}^{\infty} (k+1)^m (2^k r_i)^{-\frac{1}{2}} \right) \omega(y) \mathrm{d}y$$

$$\leq C\lambda^{-1} \sum_i \int_{B_i} |h_i(y)| \left(\sum_{k=1}^{\infty} (k+1)^m 2^{-k/2} \right) \omega(y) \mathrm{d}y \leq C\lambda^{-1} \sum_i \int_{B_i} |h_i(y)| \omega(y) \mathrm{d}y.$$

(3.12)

Next, let us consider I_{212}. Write $K(x,y,x_i) = \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i)}{|x-x_i|^{n-1}}$ for simplicity. Noting that for any $y \in B_i$, any $x \in \mathbb{R}^n \backslash 4B_i$ and t with $|x-x_i| + 2r_i \leq t$, there has $|x-y| \leq |x-x_i| + r_i < t$, then by the

cancellation condition of h_i, we have

$$I_{212} \le C\lambda^{-1} \sum_i \int_{\mathbb{R}^n \setminus 4B_i} \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right| \left(\int_{B_i} |K(x,y,x_i)||h_i(y)| \left(\int_{|x-x_i|+2r_i}^\infty \frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}} \mathrm{d}y\right) \omega(x)\mathrm{d}x$$

$$\le C\lambda^{-1} \sum_i \int_{\mathbb{R}^n \setminus 4B_i} \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right| \left(\int_{B_i} \frac{|K(x,y,x_i)||h_i(y)|}{|x-x_i|} \mathrm{d}y\right) \omega(x)\mathrm{d}x$$

$$\le C\lambda^{-1} \sum_i \int_{B_i} |h_i(y)| \sum_{k=1}^\infty (2^k r_i)^{-1} \left(\int_{2^{k+1}B_i \setminus 2^k B_i} |K(x,y,x_i)| \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right| \omega(x)\mathrm{d}x\right) \mathrm{d}y$$

By the Hölder's inequality, Lemma 2.1 and Lemma 2.5, there has

$$\int_{2^{k+1}B_i \setminus 2^k B_i} |K(x,y,x_i)| \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right| \omega(x)\mathrm{d}x$$

$$\le \left(\int_{2^{k+1}B_i \setminus 2^k B_i} |K(x,y,x_i)|^r \mathrm{d}x\right)^{1/r} \left(\int_{2^{k+1}B_i \setminus 2^k B_i} \prod_{j=1}^m \left|b_j(x) - (b_j)_{B_i}\right|^{r'} \omega^{r'}(x)\mathrm{d}x\right)^{1/r'}$$

$$\le C(k+1)^m 2^k r_i \left(2^{-k} + \int_{\frac{|y-x_i|}{2^{k+1}r_i}}^{\frac{|y-x_i|}{2^k r_i}} \frac{\omega_r(\delta)}{\delta}\mathrm{d}\delta\right) \inf_{y \in B_i} \omega(y).$$

Therefore,

$$I_{212} \le C\lambda^{-1} \sum_i \int_{B_i} |h_i(y)|\omega(y) \sum_{k=1}^\infty (k+1)^m \left(2^{-k} + \int_{\frac{|y-x_i|}{2^{k+1}r_i}}^{\frac{|y-x_i|}{2^k r_i}} \frac{\omega_r(\delta)}{\delta}\mathrm{d}\delta\right)\mathrm{d}y$$

$$\le C\lambda^{-1} \sum_i \int_{B_i} |h_i(y)|\omega(y) \left(\sum_{k=1}^\infty (k+1)^m 2^{-k} + \int_0^1 \frac{\omega_r(\delta)}{\delta}\left(\log\frac{1}{\delta}\right)^m \mathrm{d}\delta\right)\mathrm{d}y \qquad (3.13)$$

$$\le C\lambda^{-1} \sum_i \int_{B_i} |h_i(y)|\omega(y)\mathrm{d}y.$$

Note that $h_i(y) = f(y) + f_{B_i}$ when $y \in B_i$, it follows from (3.6), (3.8), (3.12) and (3.13) that

$$I_{21} \le C\lambda^{-1} \sum_i \int_{B_i} |h_i(y)|\omega(y)\mathrm{d}y \le C\lambda^{-1} \sum_i \int_{B_i} (|f(y)| + |f_{B_i}|)\omega(y)\mathrm{d}y \le C\lambda^{-1}\|f\|_{L^1_\omega(\mathbb{R}^n)}.$$

To estimate I_{23}, noting that $\Omega \in L^r(S^{n-1})$ for some $r > 1$ and $\omega^{r'} \in A_1$, using Lemma 2.2, (2.6), (2.7), Lemma 2.5, (2.5) and (3.4), we have

$$I_{23} \le \omega\left(\left\{x \in \mathbb{R}^n : \mu_\Omega\left(\sum_i h_i \prod_{j=1}^m ((b_j)_{B_i} - b_j)\right)(x) > \frac{\lambda}{6}\right\}\right)$$

$$\le C\lambda^{-1} \int_{\mathbb{R}^n} \sum_i |h_i(x)|\omega(x) \prod_{j=1}^m \left|(b_j)_{B_i} - b_j(x)\right|\mathrm{d}x$$

$$\le C\lambda^{-1} \sum_i \left(\int_{B_i} |f(x)|\omega(x) \prod_{j=1}^m \left|(b_j)_{B_i} - b_j(x)\right|\mathrm{d}x + \int_{B_i} |f_{B_i}|\omega(x) \prod_{j=1}^m \left|(b_j)_{B_i} - b_j(x)\right|\mathrm{d}x\right)$$

$$\le C\lambda^{-1} \sum_i \omega(B_i)\|f\|_{L(\log L)^m, B_i, \omega} \prod_{j=1}^m \left\|\left|b_j - (b_j)_{B_i}\right|\right\|_{\exp L, B_i, \omega}$$

$$+ C\lambda^{-1} \sum_i \frac{1}{|B_i|} \int_{B_i} |f(y)|\mathrm{d}y \int_{B_i} \omega(x) \prod_{j=1}^m \left|(b_j)_{B_i} - b_j(x)\right|\mathrm{d}x$$

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$$\leq \ C\lambda^{-1}\sum_i\Big(\omega(B_i)\|f\|_{L(\log L)^m,B_i,\omega}+\int_{B_i}|f(y)|\mathrm{d}y\inf_{y\in B_i}\omega(y)\Big)$$

$$\leq \ C\lambda^{-1}\sum_i\Big(\omega(B_i)\inf\Big\{\lambda+\frac{\lambda}{\omega(B_i)}\int_{B_i}\Phi_m\Big(\frac{|f(y)|}{\lambda}\Big)\omega(y)\mathrm{d}y\Big\}+\int_{B_i}|f(y)|\omega(y)\mathrm{d}y\Big)$$

$$\leq \ C\sum_i\Big(\omega(B_i)+\int_{B_i}\Phi_m\Big(\frac{|f(y)|}{\lambda}\Big)\omega(y)\mathrm{d}y\Big)+C\lambda^{-1}\int_{\mathbb{R}^n}|f(y)|\omega(y)\mathrm{d}y$$

$$\leq \ C\sum_i\Big(\lambda^{-1}\int_{B_i}|f(y)|\omega(y)\mathrm{d}y+\int_{B_i}\frac{|f(y)|}{\lambda}\Big(1+\log^+\frac{|f(y)|}{\lambda}\Big)^m\omega(y)\mathrm{d}y\Big)+C\lambda^{-1}\int_{\mathbb{R}^n}|f(y)|\omega(y)\mathrm{d}y$$

$$\leq \ C\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}\Big(1+\log^+\frac{|f(y)|}{\lambda}\Big)^m\omega(y)\mathrm{d}y.$$

Now, let us turn to estimate for I_{22}. Using the Minkowski's inequality, we have

$$I_{22}=\omega\Big(\Big\{x\in\mathbb{R}^n\setminus E:\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\Big|\big(b(x)-b_{B_i}\big)_\sigma\Big|\mu_\Omega\big(h_i\big(b_{B_i}-b\big)_{\sigma'}\big)(x)>\frac{\lambda}{6}\Big\}\Big)$$

$$\leq C\lambda^{-1}\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\int_{\mathbb{R}^n\setminus 4B_i}\Big|\big(b(x)-b_{B_i}\big)_\sigma\Big|\mu_\Omega\big(h_i\big(b_{B_i}-b\big)_{\sigma'}\big)(x)\omega(x)\mathrm{d}x$$

$$\leq C\lambda^{-1}\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\int_{\mathbb{R}^n\setminus 4B_i}\Big|\big(b(x)-b_{B_i}\big)_\sigma\Big|\Big(\int_0^{|x-x_i|+2r_i}\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)h_i(y)}{|x-y|^{n-1}}$$

$$\times\big(b_{B_i}-b(y)\big)_{\sigma'}\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}}\omega(x)\mathrm{d}x$$

$$+C\lambda^{-1}\sum_i\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\int_{\mathbb{R}^n\setminus 4B_i}\Big|\big(b(x)-b_{B_i}\big)_\sigma\Big|\Big(\int_{|x-x_i|+2r_i}^\infty\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)h_i(y)}{|x-y|^{n-1}}$$

$$\times\big(b_{B_i}-b(y)\big)_{\sigma'}\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}}\omega(x)\mathrm{d}x=C\lambda^{-1}\sum_i(I_{221}+I_{222}).$$

For I_{221} and I_{222}, similar to the estimates for I_{21} and I_{23}, we can get

$$I_{221}\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}r_i^{1/2}\int_{\mathbb{R}^n\setminus 4B_i}\Big|\big(b(x)-b_{B_i}\big)_\sigma\Big|\Big(\int_{B_i}\frac{|\Omega(x-y)||h_i(y)|}{|x-y|^{n+1/2}}\big(b_{B_i}-b(y)\big)_{\sigma'}\mathrm{d}y\Big)\omega(x)\mathrm{d}x$$

$$\leq C\Big(\omega(B_i)\inf\Big\{\lambda+\frac{\lambda}{\omega(B_i)}\int_{B_i}\Phi_m\Big(\frac{|f(y)|}{\lambda}\Big)\omega(y)\mathrm{d}y\Big\}+\int_{B_i}|f(y)|\omega(y)\mathrm{d}y\Big).$$

$$I_{222}\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in\mathscr{C}_j^m}\int_{\mathbb{R}^n\setminus 4B_i}\Big|\big(b(x)-b_{B_i}\big)_\sigma\Big|\Big(\int_{B_i}\frac{|K(x,y,x_i)||h_i(y)|}{|x-x_i|}\big(b_{B_i}-b(y)\big)_{\sigma'}\mathrm{d}y\Big)\omega(x)\mathrm{d}x$$

$$\leq C\Big(\omega(B_i)\inf\Big\{\lambda+\frac{\lambda}{\omega(B_i)}\int_{B_i}\Phi_m\Big(\frac{|f(y)|}{\lambda}\Big)\omega(y)\mathrm{d}y\Big\}+\int_{B_i}|f(y)|\omega(y)\mathrm{d}y\Big).$$

Thus, we have

$$I_{22}\leq C\lambda^{-1}\sum_i\Big(\omega(B_i)\inf\Big\{\lambda+\frac{\lambda}{\omega(B_i)}\int_{B_i}\Phi_m\Big(\frac{|f(y)|}{\lambda}\Big)\omega(y)\mathrm{d}y\Big\}+\int_{B_i}|f(y)|\omega(y)\mathrm{d}y\Big)$$

$$\leq C\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}\Big(1+\log^+\frac{|f(y)|}{\lambda}\Big)^m\omega(y)\mathrm{d}y.$$

From (3.7) and the above estimates for I_{21}, I_{22} and I_{23}, we have

$$I_2 \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \Big(1 + \log^+ \frac{|f(y)|}{\lambda} \Big)^m \omega(y) \mathrm{d}y.$$

This finishes the proof of Theorem 1.1. □

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